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ASYMPTOTIC METHODS IN THE AXISYMMETRIC DYNAMIC NON-STATIONARY CONTACT PROBLEM FOR AN ELASTIC HALF-SPACE†

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Asymptotic methods for solving the axisymmetric dynamic non-stationary contact problem for short and long values of the time of indentation of a rigid punch into an elastic half-space are developed. Using Laplace integral transformations (with respect to time) and Hankel integral transformations (with respect to the coordinate) the contact problem is reduced to solving an integral equation in the unknown Laplace transformant of the contact stresses under the punch. The zeroth term of the asymptotic solution of the integral equation for large values of the Laplace parameter (short times) is constructed using a special approximation in the complex plane of the symbol of the integral-equation kernel. The asymptotic solution of the integral equation for small values of the Laplace parameter (long times) is constructed in powers of this parameter. The solution of the contact problem is obtained using an inverse Laplace transformation, applied to the solutions of the integral equation.

The problem was investigated previously in [1–4], etc.

1. FORMULATION OF THE PROBLEM

We will consider the dynamic non-stationary contact problem of the indentation of a rigid punch of radius a ($r \leqslant a$) into an elastic half-space ($z \geqslant 0$, $0 \leqslant r < \infty$). The friction forces between the punch and the half-space are ignored. The form of the punch and its settlement on the half-space is given by the function $f(r, t)$($0 \leqslant r \leqslant a$, $t \geqslant 0$).

The equilibrium equations of the theory of elasticity in the case of axial symmetry of the stress-strain state of an elastic medium can be written in the form [2, 5]

$$-\frac{u}{r^2}+\beta^2\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+(1-\beta^2)\frac{\partial^2 w}{\partial r\partial z}=\frac{1}{c_1^2}\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 w}{\partial z^2}+\beta^2\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)+(1-\beta^2)\left(\frac{\partial^2 u}{\partial r\partial z}+\frac{1}{r}\frac{\partial u}{\partial z}\right)=\frac{1}{c_2^2}\frac{\partial^2 w}{\partial t^2} \tag{1.1}$$

$$\beta^2=\frac{c_2^2}{c_1^2}=\frac{\mu}{\lambda+2\mu},\quad c_1=\sqrt{\frac{\lambda+2\mu}{\rho}},\quad c_2=\sqrt{\frac{\mu}{\rho}}$$

The functions $u(r, z, t)$, $w(r, z, t)$ in (1.1) are the radial and normal displacements of the elastic medium, c_1 and c_2 are the velocities of longitudinal and transverse displacement and stress waves in the elastic medium, ρ is the density of the material of the elastic medium, and λ and μ are the Lamé elasticity constants.

At the initial instant of time, assuming that before the indentation the elastic medium is at rest, its displacements $u(r, z, t)$, $w(r, z, t)$ and their velocities $\partial u/\partial t$, $\partial w/\partial t$ are equal to zero.

The displacements of the elastic medium u and w when u, w $r, z \to \infty$ (in the rest zone) are equal to zero, together with their partial derivatives.

In the generally accepted notation of the theory of elasticity [5] the mixed boundary conditions of the problem have the form ($t \geqslant 0, z = 0$)

$$\tau_{yz}=0 \quad (0\leqslant r<\infty),\quad \sigma_z=0\quad (a<r<\infty)$$
$$w=f(r,t)\quad (0\leqslant r\leqslant a) \tag{1.2}$$

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where τ_{yz}, σ_z are the shear and normal stresses respectively on the surface of the elastic medium.

2. THE INTEGRAL EQUATION OF THE CONTACT PROBLEM

The contact problem in question reduces to solving an integral equation using the Laplace integral transformation (with respect to time t)

$$
\begin{aligned}
w^L(r,z,p) &= \int_0^\infty w(r,z,t)e^{-pt}dt \\
w(r,z,t) &= \frac{1}{2\pi i}\int_{-i\infty+c}^{i\infty+c} w^L(r,z,p)e^{pt}dp
\end{aligned}
\tag{2.1}
$$

and the Hankel integral transformation (with respect to the coordinate r)

$$
\begin{aligned}
w^{LH}(\gamma,z,p) &= \int_0^\infty w^L(r,z,p)J_0(\gamma r)rdr \\
w^L(r,z,p) &= \int_0^\infty w^{LH}(\gamma,z,p)J_0(\gamma r)\gamma d\gamma
\end{aligned}
\tag{2.2}
$$

where $J_0(r)$ is the zero-order Bessel function. These transformations are applied in succession to the differential equations of motion of the elastic medium (1.1) and to the boundary conditions of contact problem (1.2), taking the initial conditions and the conditions at infinity into account. It is assumed here that the integrals in (2.1) and (2.2) of the elastic displacement functions $u(r, z, t)$, $w(r, z, t)$, $f(r, t)$ exist. As a result of product operations, as an intermediate result, we obtain formulae for the displacement and stress transformants of the elastic medium

$$
w^L(r,z,p) = \int_0^\infty \varphi^{LH}(\gamma,p)\sigma_2[2\gamma^2e^{-\sigma_1 z} - (\gamma^2+\sigma_1^2)e^{-\sigma_2 z}]\frac{J_0(r\gamma)}{R_0(\gamma,p)}d\gamma \tag{2.3}
$$

$$
\sigma_z^L(r,z,p) = -\int_0^\infty \varphi^{LH}(\gamma,p)[4\mu\gamma^2\sigma_1\sigma_2e^{-\sigma_1 z} + (\gamma^2+\sigma_1^2)(\lambda\gamma^2-(\lambda+2\mu)\sigma_2^2)e^{-\sigma_2 z}]\frac{J_0(r\gamma)}{R_0(\gamma,p)}\gamma d\gamma \tag{2.4}
$$

in which we have introduced the following notation

$$
\begin{aligned}
&\sigma_1 = \sqrt{\gamma^2+\rho p^2/\mu}, \quad \sigma_2 = \sqrt{\gamma^2+\rho p^2/(\lambda+2\mu)} \\
&R_0(\gamma,p) = 4\mu\gamma^2\sigma_1\sigma_2 + (\gamma^2+\sigma_1^2)(\lambda\gamma^2-(\lambda+2\mu)\sigma_2^2)
\end{aligned}
\tag{2.5}
$$

and $\varphi^{LH}(\gamma, p)$ is the Laplace-Hankel transformant of the unknown contact stresses under the punch $\varphi(r, t)$, where $\sigma_z(r, 0, t) = -\varphi(r, t)$.

Realization of the mixed boundary conditions of contact problem (1.2) in the Laplace transformants, taking relation (2.3) into account, leads to the integral equation of the problem

$$
\begin{aligned}
&\int_0^a \varphi^L(\xi,p)\xi k(\xi,r)d\xi = f^L(r,p) \quad (0 \leqslant r \leqslant a) \\
&k(\xi,r) = \int_0^\infty \gamma\sigma_2(\gamma^2-\sigma_1^2)\frac{J_0(\gamma\xi)J_0(r\gamma)}{R_0(\gamma,p)}d\gamma
\end{aligned}
\tag{2.6}
$$

which is an integral equation of the first kind in the Laplace transformant of the unknown contact stresses $\varphi^L(r,p)$. For convenience later, integral equation (2.6) will be reduced to dimensionless form using the replacement of variables $r = ar'$, $\xi = a\xi'$, $\gamma = u'p/c_2$ and taking into account the notation introduced above

$$\int_0^1 \varphi^L(\xi,p)\xi k_0\left(\frac{\xi}{\Lambda},\frac{r}{\Lambda}\right)d\xi = \theta_0 f^L(r,p) \quad (0\leqslant r\leqslant 1)$$

$$k_0(\xi,r)=\int_0^\infty K(u)J_0(\xi u)J_0(ru)du,\quad K(u)=\frac{u\sigma_2}{R(u)} \tag{2.7}$$

$$R(u)=(2u^2+1)^2-4u^2\sigma_1\sigma_2,\quad \sigma_1=\sqrt{u^2+1},\quad \sigma_2=\sqrt{u^2+\beta^2}$$

$$\theta_0=\mu c_2/(a^2p),\quad \Lambda=c_2/(pa)$$

Integral equation (2.7) then reduces to the equivalent integral equation with a difference kernel [6]. To solve it we multiply the left and right-hand sides of (2.7) by

$$\frac{rdr}{\sqrt{x^2-r^2}}$$

and then integrate with respect to r from 0 to x. After interchanging the order of integration, using integral representations of the form [7]

$$\int_0^x \frac{rJ_0(\gamma r)}{\sqrt{x^2-r^2}}dr=\frac{\sin\gamma x}{\gamma},\quad \int_x^\infty \frac{rJ_0(\gamma r)}{\sqrt{r^2-x^2}}dr=\frac{\cos\gamma x}{\gamma} \tag{2.8}$$

and differentiating the left- and right-hand sides of the relation obtained we arrive at the integral equation

$$\int_0^1 \varphi^L(\xi,p)\xi d\xi\int_0^\infty K(\gamma)J_0\left(u\frac{\xi}{\Lambda}\right)\cos u\frac{x}{\Lambda}du=\theta_0 g(x,p) \quad (0\leqslant x\leqslant 1)$$

$$g(x,p)=\frac{d}{dx}\int_0^x \frac{rf^L(r,p)}{\sqrt{x^2-r^2}}dr \tag{2.9}$$

Using the integral representation for $J_0(x)$

$$J_0(xt)=\frac{2}{\pi}\int_0^x \frac{\cos(t\zeta)}{\sqrt{x^2-\zeta^2}}d\zeta$$

and interchanging the order of integration with respect to γ and ξ in (2.9), we obtain the new integral equation

$$\int_0^1 \omega(\xi,p)d\xi\int_0^\infty K(u)\cos u\frac{\xi}{\Lambda}\cos u\frac{x}{\Lambda}du=\frac{\pi\theta_0}{2}g(x,p) \quad (0\leqslant x\leqslant 1)$$

Continuing $\omega(x,p)$ and $g(x,p)$ evenly with respect to x in the last integral equation in the section [–1, 0], and the inner integral in the section $(-\infty,\infty)$ and multiplying it by $2(1-\beta^2)$, we obtain a convolution-type integral equation of the first kind with difference kernel

$$\int_{-1}^1 \omega(\xi,p)d\xi\int_0^\infty K(u)\cos u\frac{\xi-x}{\Lambda}du=\pi\theta g(x,p) \quad (|x|\leqslant 1) \tag{2.10}$$

in which

$$K(u)=2(1-\beta^2)|u|\sqrt{u^2+\beta^2}[R(u)]^{-1},\quad \theta=2(1-\beta^2)\theta_0 \tag{2.11}$$

$$\omega(\zeta,p)=\int_\zeta^1 \frac{\varphi^L(\xi,p)\xi}{\sqrt{\xi^2-\zeta^2}}d\xi \tag{2.12}$$

while $R(u)$ is given in (2.7).

Relation (2.12) is an Abel-type integral equation. Solving Eq.(2.12), we obtain a relation between the functions $\varphi^{L}(x,p)$ and $\omega(\zeta,p)$, expressed by the formula

$$\varphi^{L}(x,p)=-\frac{2}{\pi x}\frac{d}{dx}\int_{x}^{1}\frac{\omega(\zeta,p)\zeta}{\sqrt{\zeta^{2}-x^{2}}}d\zeta \tag{2.13}$$

3. THE ASYMPTOTIC SOLUTION OF INTEGRAL EQUATION (2.10) FOR LARGE VALUES OF p

Below, when constructing the zeroth term of the asymptotic form of the solution of integral equation (2.10), it will be necessary to factorize the generalized function $K(u)$ (2.11) in the complex plane $u = \sigma + i\tau$. To simplify this operation we replace $K(u)$ in (2.10) by the new function

$$K_{\varepsilon}(u)=2(1-\beta^{2})\sqrt{(u^{2}+\varepsilon^{2})(u^{2}+\beta^{2})}[R(u)]^{-1} \tag{3.1}$$

for which the following limit obviously holds

$$K(u)=\lim_{\varepsilon\to 0}K_{\varepsilon}(u) \tag{3.2}$$

Here, integral equation (2.1) with the new symbol of the kernel $K_{\varepsilon}(u)$ is written in the form

$$\begin{gathered}\int_{-1}^{1}\omega_{\varepsilon}(\xi,p)k_{\varepsilon}\left(\frac{\xi-x}{\Lambda}\right)d\xi=2\pi\theta g(x,p)\quad(|x|\leqslant 1)\\ k_{\varepsilon}(t)=\int_{-\infty}^{\infty}K_{\varepsilon}(u)e^{iut}du\end{gathered} \tag{3.3}$$

Since the integral operator in (3.3) tends uniformly with respect to ε in the interval $|x| \leqslant 1$ to the operator (2.10) on the left-hand side of (2.10), we have

$$\omega(x,p)=\lim_{\varepsilon\to 0}\omega_{\varepsilon}(x,p) \tag{3.4}$$

It should be noted that $K_{\varepsilon}(u)$ is an even function, which is real on the real axis of the complex plane $u = \sigma + i\tau$, and has six branching points of the algebraic type in this plane $u = \pm i$, $u = \pm i\beta$, $u = \pm i\varepsilon$ and two poles $u = \pm i\eta_0$ (Rayleigh poles); in addition, it possesses the following asymptotic properties

$$K_{\varepsilon}(u)=A\varepsilon+O(u^{2})\quad(u\to 0,\ A=2\beta(1-\beta^{2})) \tag{3.5}$$

$$K_{\varepsilon}(u)=1+O(u^{-2})\quad(u\to\infty) \tag{3.6}$$

To obtain the principal term of the asymptotic form of the solution of integral equation (3.3) for small values of the parameter Λ (large values of p) we will approximate the symbol of the integral-equation kernel $K_{\varepsilon}(u)$ by an expression $K_{\varepsilon}^{0}(u)$ of the following form

$$\begin{gathered}K_{\varepsilon}^{0}(u)=\frac{\sqrt{u^{2}+\varepsilon^{2}}\sqrt{u^{2}+\beta^{2}}}{u^{2}+\eta_{0}^{2}}M(u)\\ M(u)=\exp[-d(\sqrt{\beta+iu}\sqrt{1+iu}+\sqrt{\beta-iu}\sqrt{1-iu}-1-\beta)]\end{gathered} \tag{3.7}$$

in the complex plane $u = \sigma + i\tau$, where the approximation parameter d is found from the condition $K_{\varepsilon}(0) = K_{\varepsilon}^{0}(0)$ and is calculated from the formula

$$d = (1-\sqrt{\beta})^{-2}\ln(A\eta_0^2/\beta) \tag{3.8}$$

The approximation $K_\varepsilon^0(u)$ possesses all the above-mentioned properties of the function $K_\varepsilon(u)$ in the complex plane and is single-valued and regular in this plane with cuts drawn in it from the branching points to $\pm\infty$ along the imaginary axis ($\sigma = 0$) with knocked-out points $u = \pm i\eta_0$ and chosen branches on the cuts with the condition that $\sqrt{1} = 1$.

Moreover, along the real axis ($\tau = 0$) of the complex plane, $K_\varepsilon(u)$ and $K_\varepsilon^0(u)$ are close to one another, and the difference between them for all $\nu \in [0, 0, 44]$ does not exceed 4%. Note that an approximation of the form (3.7) is not unique, but may be the simplest.

The principal term of the asymptotic form of the solution of integral equation (3.3) for small values of Λ can be represented in multiplicative form [8, 9] by the formula

$$\omega_\varepsilon(x,p) = \mu_0\omega_+^\varepsilon\left(\frac{1+x}{\Lambda},p\right)\omega_-^\varepsilon\left(\frac{1-x}{\Lambda},p\right)\left[\omega_\infty^\varepsilon\left(\frac{x}{\Lambda},\ p\right)\right]^{-1} \tag{3.9}$$

in which the functions $\omega_\pm^\varepsilon(x,p)$ and $\omega_\infty{}^\varepsilon(x,p)$ are solutions of the integral equations

$$\int_0^\infty \omega_\pm^\varepsilon(\xi,p)d\xi\int_{-\infty}^\infty K_\varepsilon^0(u)e^{iu(\xi-x)}du = \frac{2\pi\theta}{\Lambda}g(\pm\Lambda x\mp 1,p) \tag{3.10}$$

$$(0 < x < \infty)$$

$$\int_{-\infty}^\infty \omega_\infty^\varepsilon(\xi,p)d\xi\int_{-\infty}^\infty K_\varepsilon^0(u)e^{iu(\xi-x)}du = \frac{2\pi\theta}{\Lambda}g(\Lambda x,p) \tag{3.11}$$

$$(-\infty < x < \infty)$$

where μ_0 is a correction factor, found when solving integral equation (3.3).

The solution of integral equation (3.10) in the case of a plane punch $g(x,p) = f^L(p)$ can be fairly simply determined by the Wiener-Hopf method [9, 10] and is given by the formula

$$\omega_\pm^\varepsilon(x,p) = -\frac{\theta f^L(p)}{\Lambda K_{\varepsilon-}^0(0)}\frac{1}{2\pi}\int_{-\infty}^\infty \frac{e^{-iux}}{iuK_{\varepsilon+}^0(u)}du \tag{3.12}$$

In (3.12) the functions are regular functions in the upper half-plane (Im $(u) > -\varepsilon$, $\varepsilon > 0$) and the lower half-plane (Im $(u) < \varepsilon$), respectively, and satisfy the factorization condition

$$K_\varepsilon^0(u) = K_{\varepsilon+}^0(u)K_{\varepsilon-}^0(u)$$

in the complex plane $u = \sigma + i\tau$ c with the above-mentioned cuts. The general form of these functions is given by the formula

$$K_{\pm\varepsilon}^0(u) = \frac{\sqrt{\varepsilon\mp iu}\sqrt{\beta\mp iu}}{\eta_0\mp iu}M_\pm(u)$$
$$M_\pm(u) = \exp\left[-d\left(\sqrt{\beta\mp iu}\sqrt{1\mp iu}\pm iu-\frac{1+\beta}{2}\right)\right] \tag{3.13}$$

The solution of Eq.(3.11) in this case – the case of a plane punch – is given by the formula [9]

$$\omega_\infty^\varepsilon(x,p) = \frac{\theta}{2\pi\Lambda}\frac{f^L(p)}{K_\varepsilon^0(0)}\quad(-\infty < x < \infty) \tag{3.14}$$

The most promising form of the solution of integral equation (3.10) for further use is the solution which is represented in the form of certain integrals, obtained by calculating the contour integral in (3.12)

$$\omega_{\pm}^{\varepsilon}(x,p)=\frac{\theta f^{L}(p)}{\Lambda K_{\varepsilon-}^{0}(0)}\Omega_{\pm}^{\varepsilon}(x,p) \tag{3.15}$$

$$\Omega_{\pm}^{\varepsilon}(x,p)=\frac{1}{\pi}\int_{\beta}^{1}\vartheta(y)e^{-d(1+\beta-2y)/2}\sin(d\sqrt{1-y}\sqrt{y-\beta})e^{-yx}dy+$$

$$+\frac{1}{\pi}\int_{\varepsilon}^{\beta}\vartheta(y)e^{-d(\sqrt{1-y}-\sqrt{\beta-y})^{2}/2}e^{-yx}dy+\frac{1}{K_{\varepsilon-}^{0}(0)}$$

$$\vartheta(y)=\frac{y-\eta_{0}}{y\sqrt{y-\varepsilon}\sqrt{\pm\beta\mp y}},\quad K_{\varepsilon-}^{0}(0)=\frac{\sqrt{\varepsilon}\sqrt{\beta}}{\eta_{0}}e^{d(1-\sqrt{\beta})^{2}/2}$$

Taking the notation introduced and formula (3.15) and (3.14) into account, the solution of integral equation (3.3) in the form (3.9) can be written as

$$\omega_{\varepsilon}(x,p)=2\mu_{0}(1-\beta^{2})\frac{\mu}{a}f^{L}(p)\Omega_{+}^{\varepsilon}\left(\frac{1+x}{\Lambda},p\right)\Omega_{-}^{\varepsilon}\left(\frac{1-x}{\Lambda},p\right) \tag{3.16}$$

For the final representation of the function $\omega_{\varepsilon}(x,p)$ it is necessary to determine the correction factor μ_0 in (3.16), which is determined [11,12] by substituting expression (3.16) with $\varepsilon = 0$ into Eq.(3.3), as a result of which we obtain the equation

$$2\mu_{0}(1-\beta^{2})\frac{\mu}{a}f^{L}(p)\int_{-1}^{1}\Omega_{+}^{0}\left(\frac{1+\xi}{\Lambda},p\right)\Omega_{-}^{0}\left(\frac{1-\xi}{\Lambda},p\right)k_{0}\left(\frac{\xi-x}{\Lambda}\right)d\xi=2\pi\theta f^{L}(p) \tag{3.17}$$

To evaluate the integral in (3.17) approximately for small Λ, we establish the principal terms of the asymptotic form of the functions occurring in the integrand as $\Lambda\to 0$. We have

$$\Omega_{\pm}^{0}\left(\frac{1\pm x}{\Lambda},p\right)=\frac{2}{\sqrt{\pi A}}\sqrt{\frac{1\pm x}{\Lambda}} \tag{3.18}$$

$$k_{0}\left(\frac{\xi-x}{\Lambda}\right)=2\int_{0}^{\infty}K_{0}(u)\cos u\frac{\xi-x}{\Lambda}du=-2(1-\beta^{2})\beta\Lambda^{2}\frac{d}{dx}\frac{1}{\xi-x} \tag{3.19}$$

When obtaining (3.19) we used the integral

$$\int_{0}^{\infty}u\cos u(\xi-x)du=-\frac{d}{dx}\frac{1}{\xi-x}$$

which is understood in the sense of the generalized functions of the theory of the Fourier integral transformation [13]. Substituting (3.18) and (3.19) into (3.17) and comparing its left- and right-hand sides, we obtain that

$$\mu_{0}=\pi/4 \tag{3.20}$$

Transferring now to the Laplace originals in (3.16) for $\omega_{\varepsilon}(x,p)$ and then taking the limit in the formulae obtained as $\varepsilon\to 0$, we obtain

$$\omega(x,t)=\frac{\pi}{2}(1-\beta^{2})\left[\frac{\mu}{a}\int_{0}^{t}f(\tau)E(x,t-\tau)d\tau+f_{0}E(x,t)\right] \tag{3.21}$$

$$E(x,t)=\int_{0}^{t}\Omega_{+}^{0}(a+x,\tau)\Omega_{-}^{0}(a-x,t-\tau)d\tau$$

$$\Omega_{\pm}^{0}(x,t)=\lim_{\varepsilon\to 0}\Omega_{\pm}^{\varepsilon}(x,t)$$

$$\Omega_{\pm}^{0}(a\pm x,t)=\frac{\eta(t)e(0,t_1^{\pm})}{\sqrt{t_1^{\pm}-t}}\exp\left[-\frac{d}{2}\frac{(\sqrt{t_2^{\pm}-t}-\sqrt{t_1^{\pm}-t})^2}{t_2^{\pm}}\right]+$$

$$+\frac{\eta(t)e(t_1^{\pm},t_2^{\pm})}{\sqrt{t-t_1^{\pm}}}\exp\left(-\frac{1+\beta}{2}d+\frac{dt}{t_2^{\pm}}\right)\sin d\frac{\sqrt{t_2^{\pm}-t}\sqrt{t-t_1^{\pm}}}{t_2^{\pm}}$$

$$\eta(t)=\frac{t-\eta_0 t_2^{\pm}}{\pi t\sqrt{t}},\quad e(u,v)=H(t-u)-H(t-v),\qquad t_i^{\pm}=\frac{a\pm x}{c_i}\ (i=1,2)$$

$H(t)$ is the Heaviside function an f_0 is the initial indentation of the punch prior to the instant of time $t = 0$. The asymptotic solution of the contact problem for small t is given by the formula obtained after applying an inverse Laplace transformation to (2.12)

$$\varphi(r,t)=\frac{2a}{\pi}\left[\frac{\omega(a,t)}{\sqrt{a^2-r^2}}-\int_r^a\frac{\omega'_{\xi}(\xi,t)}{\sqrt{\xi^2-r^2}}d\xi\right]\quad(0\leqslant r\leqslant a) \tag{3.22}$$

and which holds for $t < 2a/c_2$, since the asymptotic solution of integral equation (3.3) in the form (3.9) is affected when $\Lambda = c_2/ap < 2$ [11].

Analysis of formulae (3.22) and (3.21), which determine the field of the contact stresses $\varphi(r, t)$, shows that it has a fixed root-type singularity and a mobile singularity of the same type on the front of the longitudinal wave which propagates from the edge of the punch to its centre. There is no singularity at the front of the transverse wave, which propagates under the punch after the longitudinal wave.

4. THE ASYMPTOTIC SOLUTION OF INTEGRAL EQUATION (2.10) FOR SMALL VALUES OF p

Based on the asymptotic properties of the symbol of the kernel of integral equation (2.10) (the function $K(u)$), defined by (3.5) and (3.6), the kernel of this equation can be represented in the form

$$\frac{1}{2}k(t)=\pi\delta(t)-\int_0^\infty(1-K(u))\cos ut\,du \tag{4.1}$$

where $\delta(t)$ is the Dirac delta function. This representation of the kernel enables integral equation (2.10) to be written in the form of an integral equation of the second kind

$$\omega(x,p)=\frac{\theta}{\Lambda}g(x,p)+\frac{1}{\pi\Lambda}\int_{-1}^{1}\omega(\xi,p)F\left(\frac{\xi-x}{\Lambda}\right)d\xi$$
$$F(t)=\int_0^\infty(1-K(u))\cos ut\,du \tag{4.2}$$

in which the function $F(t)$ is even and is represented in the form

$$F(t)=\sum_{n=0}^{4}a_n|t|^n+F_1(t) \tag{4.3}$$

$$a_0=\int_0^\infty(1-K(u))du,\quad a_1=\frac{\pi}{2}h_1,\quad a_2=-\frac{1}{2}\int_0^\infty(u^2-u^2K(u)+h_1)du$$

$$a_3=-\frac{\pi}{12}h_2,\quad a_4=\frac{1}{24}\int_0^\infty(u^4-u^4K(u)+h_1u^2+h_2)du$$

$$F_1(t)=\int_0^\infty\left(1-K(u)+\frac{h_1}{u^2}+\frac{h_2}{u^4}\right)\left(\cos ut-1+\frac{u^2t^2}{2!}-\frac{u^4t^4}{4!}\right)du=O(t^6)$$

The estimate in (4.3) for the function $F_1(t)$ follows from the representation of $K(u)$ for large values of u

$$K(u)=1+\frac{h_1}{u^2}+\frac{h_2}{u^4}+O\left(\frac{1}{u^6}\right) \quad (u\to\infty) \tag{4.4}$$

$$h_1=-\frac{3\beta^4-4\beta^2+3}{4(1-\beta^2)}, \quad h_2=-\frac{\beta^8+2\beta^6-18\beta^4+22\beta^2-11}{16(1-\beta^2)^2}$$

The structure of the function $F(t)$ and, consequently, of the kernel $k(t)$ (4.1), for small values of $|t|$, shows that the solution of integral equation (4.2) must be sought in the form of a section of a series in negative powers of Λ

$$\omega(x,p)=\sum_{n=0}^{3}\omega_n(x,p)\Lambda^{-n} \tag{4.5}$$

To determine the terms of the series $\omega_n(x,p)$ from (4.5) it is necessary to substitute (4.5) into (4.2) and equate coefficients on the left-and right-hand sides of (4.2) for like powers of Λ, as a result of which we obtain ($\theta = b\Lambda$, $b = 2(1-\beta^2)\mu a^{-1}$)

$$\begin{aligned}&\omega_0(x,p)=g(x,p)\\ &\omega_n(x,p)=\frac{1}{\pi}\sum_{k=0}^{n-1}a_k\int_{-1}^{1}\omega_{n-k}(\xi,p)|\xi-x|^n\,d\xi \quad (n\geqslant 1)\end{aligned} \tag{4.6}$$

The function $g(x,p)$ is given by the formula from (2.9). Calculating $\omega_n(x,p)$ and the quadratures, which arise here, in succession for $n = 0, 1, 2, 3$ we obtain

$$\omega_0(x,p)=b\frac{d}{dx}\int_0^x\frac{rf^L(r,p)}{\sqrt{x^2-r^2}}dr, \quad \omega_1(x,p)=\frac{2ba_0}{\pi}\int_0^1\frac{rf^L(r,p)}{\sqrt{1-r^2}}dr$$

$$\begin{aligned}\omega_2(x,p)=\frac{2b}{\pi}\Bigg[&\int_0^1\frac{rf^L(r,p)}{\sqrt{1-r^2}}\left(\frac{2a_0^2}{\pi}+a_1(1-\sqrt{1-r^2}\,l(1,r))\right)dr+\\ &+a_1\int_0^x rf^L(r,p)l(x,r)dr\Bigg]\end{aligned} \tag{4.7}$$

$$\begin{aligned}\omega_3(x,p)=\frac{2b}{\pi}\Bigg[&\int_0^1\frac{rf^L(r,p)}{\sqrt{1-r^2}}\left(\frac{4a_0^2}{\pi^2}+\frac{2a_0a_1}{\pi}(1-\sqrt{1-r^2}\,l(1,r))\right)dr+\\ &+\int_0^1 rf^L(r,p)(2a_1l(1,r)-(a_1+2a_2)\sqrt{1-r^2})dr+\\ &+(1+x^2)\left(\frac{a_0a_1}{\pi}+a_2\right)\int_0^1\frac{rf^L(r,p)}{\sqrt{1-r^2}}dr\Bigg]; \quad l(x,r)=\ln\frac{x+\sqrt{x^2-r^2}}{r}\end{aligned}$$

To calculate the quadratures in (4.6) it was assumed that the function $f^L(r,p)$ was continued oddly with respect to r in the section $[-1, 0]$. Formulae (4.5) and (4.7) give the asymptotic solution of integral equation (4.2) or, which is the same thing, Eq.(3.1), for large values of Λ (or small p).

To obtain the asymptotic solution of the contact problem in question for large t, which is given by (3.22) in dimensional variables r and t, it is necessary to transfer in (4.5) to the Laplace originals for determining $\omega(x,t)$. Formally making this transition, in the general case of $f(r,t)$, we obtain the solution of the contact problem in the form

$$\varphi(r,t)=\frac{2a}{\pi}\sum_{n=0}^{\infty}\left(\frac{a}{c_2}\right)^n\left[\frac{w_n(t)}{\sqrt{a^2-r^2}}+\frac{\partial^n}{\partial t^n}\int_r^a\frac{\partial\omega_n(\zeta,t)}{\partial\zeta}\frac{\partial\zeta}{\sqrt{\zeta^2-r^2}}\right], \quad w_n(t)=\omega_n^{(n)}(a,t) \tag{4.8}$$

In formula (4.8) $\omega_n^{(k)}(a, t)$ denotes, in symbolic form, the derivative with respect to t of the functions $\omega_n(a, t)$. The functions $\omega_n(r, t)$ are given by (4.7) ($n = 0, 1, 2, 3$), in which, after transferring to the originals, it is sufficient to replace $\omega_n(r, p)$ on the right-hand by $\omega_n(r, t)$, while the function $f^L(r, p)$ is replaced by its original $f(r, t)$.

Note that solution (4.8) holds for large t ($t > 2a/c_2$), which follows from the joining of the asymptotic solutions along the integral characteristic $P(t)$, which are calculated for short and long t in Section 5.

In the important special case when the indenting punch is plane, i.e. when $f(r, t) = f(t)$, the solution of the contact problem, ignoring the generalized component of the solution and retaining the first four terms in it, has the form

$$\varphi(r,t)=\frac{2ba}{\pi\sqrt{a^2-r^2}}\left[f(t)+\frac{2a_0}{\pi}\frac{a}{c_2}\dot{f}(t)+\right.$$
$$+\frac{2}{\pi}\left[\frac{2a_0^2}{\pi}+a_1\left(4-3\frac{r^2}{a^2}\right)\right]\left(\frac{a}{c_2}\right)^2\ddot{f}(t)+ \tag{4.9}$$
$$\left.+\frac{2}{\pi}\left[\frac{4a_0^3}{\pi^2}+\frac{3a_0a_1}{\pi}\left(3-2\frac{r^2}{a^2}\right)+\frac{2a_1}{3}+\frac{2a_2}{3}\left(11-9\frac{r^2}{a^2}\right)\right]\left(\frac{a}{c_2}\right)^3\dddot{f}(t)\right]$$

The dots denote derivatives with respect to t.

5. FORMULAE FOR CALCULATING THE FORCE $P(t)$ ACTING ON THE PUNCH

An important factor in solving the contact problem is calculating the force $P(t)$ acting on the indenting punch (or, with the opposite sign, the reaction force of the elastic medium on the punch penetrating into it). To calculate the force $P(t)$ we will use the general formula which, for the problem in question, has the form

$$P(t)=2\pi\int_0^a\varphi(r,t)r\,dr \tag{5.1}$$

and in Laplace transforms in dimensionless form has the representation

$$P^L(p)=4a^2\int_0^1\omega(\xi,p)d\xi \tag{5.2}$$

We have taken formula (2.13) for $\varphi^L(x, p)$ into account here.

We represent the function $\omega(x, p)$ by formula (3.17) with $\varepsilon = 0$ and, substituting it into (5.2), we obtain

$$P^L(p)=\frac{\pi^2ba^2}{\beta}f^L(p)\int_0^1\Omega_+^0\left(\frac{1+\xi}{\Lambda},p\right)\Omega_-^0\left(\frac{1-\xi}{\Lambda},p\right)d\xi \tag{5.3}$$

The quantity b is given before formula (4.6).

We make the following replacement of variable in the integral in (5.3)

$$1+\xi=\Lambda\xi',\quad d\xi=\Lambda d\xi' \tag{5.4}$$

and, omitting the primes, we obtain

$$P^L(p)=\frac{\pi^2ba^2}{\beta\gamma}f^L(p)\int_0^\gamma\Omega_+^0(\xi,p)\Omega_-^0(\gamma-\xi,p)d\xi\quad\left(\gamma=\frac{2}{\Lambda}\right) \tag{5.5}$$

Considering the integral in the last formula as a convolution of the Laplace functions $\Omega_{\pm}^{0}(x,p)$ and taking into account their integral representation in the form of the Laplace integral

$$\Omega_{\pm}^{0}(x,p)=\frac{1}{2\pi i}\int_{-i\infty+c}^{i\infty+c}\frac{e^{sx}}{sK_{0+}^{0}(is)}ds \quad (\operatorname{Re} c>0)$$

where $K_{0+}^{0}(u) = \lim K_{\varepsilon+}^{0}(u)$ $(\varepsilon \to 0)$, we obtain for $P^L(p)$ the principal term of the asymptotic form for small Λ in the form

$$\begin{aligned}
P^L(p)&=\frac{\pi ba^2}{\beta\gamma}f^L(p)\frac{1}{2\pi i}\int_{-i\infty+c}^{i\infty+c}\frac{e^{\gamma z}dz}{z^2[K_{0+}^{0}(iz)]^2}\\
K_{0+}^{0}(iz)&=\frac{\sqrt{z}\sqrt{\beta+z}}{\eta_0+z}\exp\left[-d\left(\sqrt{\beta+z}\sqrt{1+z}-z-\frac{1+\beta}{2}\right)\right]
\end{aligned} \tag{5.6}$$

The integrand in the complex z plane has the following singular points: two branching points $z = -\beta$ and $z = -1$, a single pole at the point $z = -\beta$ and a triple pole at the point $z = 0$. Closing the integration contour in the left half-plane with cuts drawn in it from the branching points $z = -\beta$ and $z = -1$ to $-\infty$ along the negative part of the real axis with a choice of the branches of the multivalued functions, to obtain the asymptotic formula $P(t)$ $(t < 2a/c_2)$ when evaluating integral (5.6) it is sufficient to take the triple pole at zero $(z = 0)$ into account. Converting the formula for $P^L(p)$ obtained in this way to the Laplace originals we have

$$\begin{aligned}
P(t)&=\pi^2ba^2\left[\frac{2a\dot{f}(t)}{c_2K_{1+}^2(0)}-\frac{4iK_{1+}'(0)}{K_{1+}^3(0)}f(t)-\right.\\
&\left.-\left(\frac{6K_{1+}'^2(0)}{K_{1+}^4(0)}-\frac{2K_{1+}''(0)}{K_{1+}^3(0)}\right)\frac{c_2}{2a}\int_0^t f(\tau)d\tau+\frac{2a}{c_2}\frac{f_0}{K_{1+}^2(0)}\delta(t)\right]\\
K_{1+}(iz)&=K_{0+}^{0}(iz)/z\\
K_{1+}(0)&=\sqrt{A},\quad K_{1+}'(0)=\frac{i\sqrt{A}}{2\beta\eta_0}\left(\eta_0-2\beta-\sqrt{\beta}\eta_0\ln\frac{A\eta_0^2}{\beta}\right)\\
K_{1+}''(0)&=\sqrt{A}\left[\frac{\eta_0^2+4\beta\eta_0-8\beta^2}{4\beta^2\eta_0^2}+\frac{\eta_0-2\beta}{2\beta\sqrt{\beta}\eta_0}dh^2-\frac{h^4d^2}{4\beta}-\frac{(1-\beta)^2d}{4\beta\sqrt{\beta}}\right]\\
&(h=1-\sqrt{\beta})
\end{aligned} \tag{5.7}$$

Here f_0 is the initial indentation of the punch up to the instant of time $t = 0$, the quantity A is given in (3.5) and d is given in (3.8).

In the case of long times t we obtain a formula for $P(t)$ from the formulae in Section 4. For this purpose, changing to Laplace originals in (5.2) and taking formula (4.5) for $\omega(x,p)$ into account, we obtain in the general case $(t > 2a/c_2)$

$$P(t)=4a^2\sum_{n=0}^{\infty}\left(\frac{a}{c_2}\right)^n\int_0^1\frac{\partial^n\omega_n(\zeta,t)}{\partial t^n}d\zeta \tag{5.8}$$

In the case of a plane punch $f(r, t) = f(t)$ we obtain an expression for $P(t)$ in terms of the function $f(t)$ and its derivatives $f^{(n)}(t)$

$$P(t)=\frac{4}{\pi}ba^2\sum_{n=0}^{\infty}\left(\frac{a}{c_2}\right)^n d_nf^{(n)}(t) \tag{5.9}$$

The first of the d_n $(n = 0, 1, 2, 3)$ are given by the formulae

$$d_0 = \pi,\ d_1 = 2a_0,\ d_2 = 2(2\pi^{-1}a_0^2 + 2a_1/3)$$
$$d_3 = 2\pi^{-2}(4a_0^3 + 7\pi a_0 a_1/3 + 2\pi^2 a_1/3 - 2\pi^2 a_2/3)$$

where b is defined just before formula (4.6).

In the case of the instantaneous indentation of a plane punch, its law of motion is given by the function $f(t) = f_0 H(t)$, where $H(t)$ is the Heaviside function and f_0 is the value of the indentation of the punch into the elastic half-space. It can be seen from (5.9) that in this case, when $t \gg 2a/c_2$ the value of the force $P(t)$ restraining the punch at a depth f_0, is identical asymptotically with the value of the force P corresponding to the static axisymmetric contact problem of the indentation of a plane punch into an elastic half-space [5].

A comparison of the contact stresses under the punch for short and long times of its penetration into the elastic medium is difficult in view of the fact that, in the contact stresses obtained for long times, the diffraction pattern of the wave process is not isolated in explicit form. It is represented in this solution in the form of a series in powers of the time t. Hence, a comparison of the asymptotic solutions obtained for short and long times is made with respect to the smoother characteristic of the contact problem – with respect to the force $P(t)$ acting on a plane punch when $f(t) = v_0 t$ (v_0 is the rate of penetration of the punch).

We give below values of $\omega P(t)$ ($\omega = c_2(\mu a^2 v_0)^{-1}$) for short and long times, calculated from (5.7) and (5.9) respectively for various values of $\tau = tc_1/a$

τ	0.8	1.0	1.2	1.4	1.6	1.8	
$\omega P(t)$	11.03	9.51	7.99	–	–	–	($\nu = 0.25$)
$\omega P(t)$	–	–	7.80	8.42	9.04	9.65	($\nu = 0.25$)
$\omega P(t)$	12.69	11.25	9.80	8.36	–	–	($\nu = 0.30$)
$\omega P(t)$	–	–	–	8.65	9.26	9.87	($\nu = 0.30$)

It can be seen that joining of the values of the force $P(t)$ occurs for $\nu = 0.25$ when $\tau = 1.2$, and for $\nu = 0.30$ when $\tau = 1.4$.

6. THE MOTION OF A PUNCH ON THE SURFACE OF AN ELASTIC MEDIUM

To determine the law of vertical motions of a rigid punch of mass M on the surface of an elastic half-space, we set up the differential equation of its motion like the motion of a point mass.

$$M\ddot{f}(t) = Q(t) \tag{6.1}$$

($Q(t)$ is the elastic resistance force of the medium), with initial conditions $f(0) = f_0$ (f_0 is the initial indentation of the punch up to the instant of time $t = 0$), and $\dot{f}(0) = v_0$ (v_0 is the initial rate of penetration of the punch when $t = 0$).

Applying a Laplace transformation to Eq.(6.1) we obtain the equation in terms of transformants

$$M[p^2 f^L(p) - f_0 p - v_0] = Q^L(p) \tag{6.2}$$

The transformant of the elastic resistance force of the medium $Q^L(p)$ is determined, if we take into account that $Q(t)$ is identical in value with $P(t)$ – by the force acting on the punch taken with the opposite sign

$$Q^L(p) = -P^L(p) = -\int_{-a}^{a} \varphi^L(x,p)dx \tag{6.3}$$

and, by (5.9), can be represented, for example, for small p, by the formula

$$Q^L(p) = -\frac{4}{\pi}ba^2 \sum_{n=0}^{\infty} (hp)^n d_n f^L(p) \quad \left(h = \frac{a}{c_2}\right) \tag{6.4}$$

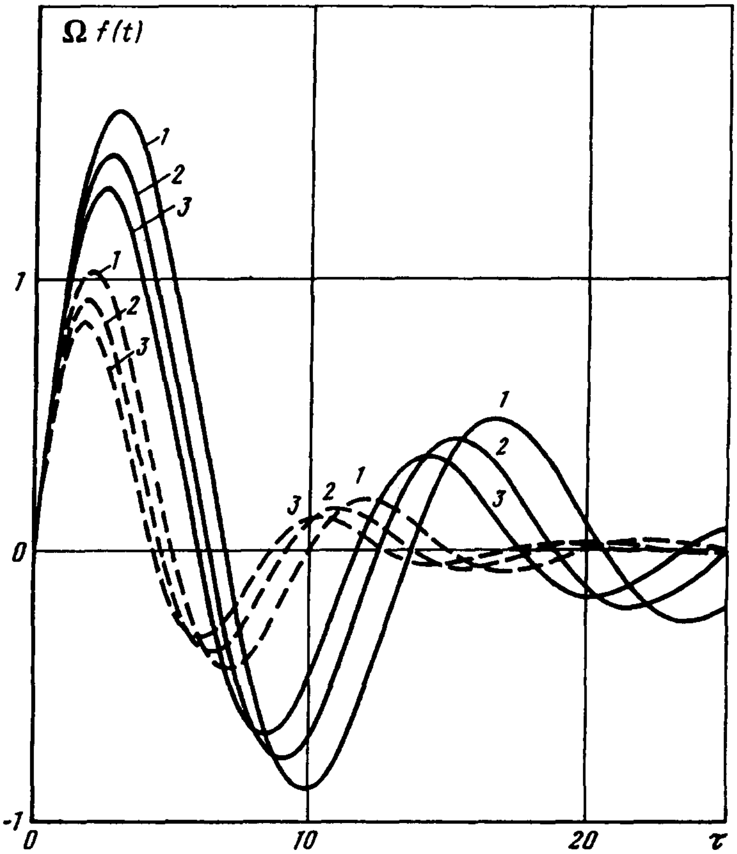

Fig. 1.

Substituting (6.4) into (6.2) and retaining terms up to $(hp)^2$, we determine the transformant $f^L(p)$ $(f_0 = 0)$

$$f^L(p) = \frac{Mv_0}{8(1-\beta^2)\mu a}\frac{1}{1+\eta_1 hp + \eta_2 (hp)^2}$$
$$\eta_1 = \pi d_1, \quad \eta_2 = (\eta h^2)^{-1} + d_2 \tag{6.5}$$

Transferring to the Laplace originals in (6.5), we obtain for large values of t $(t > 2h)$

$$\Omega f(t) = \frac{1}{\omega\alpha}\exp\left(-\frac{\eta_1}{2\eta_2}\tau\right)\sin\omega\tau, \quad \Omega = \frac{\mu h}{\rho a^2 v_0}, \quad \tau = \frac{t}{h}$$
$$\omega^2 = \frac{4\eta_2 - \eta_1^2}{4\eta_2^2}, \quad \alpha = 1 + \frac{2}{\pi}\left(\frac{2a_0^2}{\pi} + \frac{2}{3}a_1\right)\eta h^2, \quad \eta = \frac{8(1-\beta^2)\mu a}{M} \tag{6.6}$$

Graphs of $\Omega f(t)$ as a function of τ for various values of the parameter $m = \rho/(\pi\rho_0)$ $(\rho_0 = M/(\pi a^3))$ and Poisson's ratio ν are shown in Fig. 1. Curves 1, 2 and 3 correspond to $\nu = 0.10$, 0.25 and 0.35 respectively. The continuous curves correspond to $m = 0.1$ and the dashed curves correspond to $m = 0.05$. It can be seen that the parameter m, which represents the change in the mass of the punch M for fixed ρ, has a considerable influence on the value of the amplitude of the vibrations of the punch and their attenuation, and also on the value of the oscillation frequency of the punch on the elastic half-space.

For small values of t ($t < 2h$), to determine the law of motion of the punch, it is sufficient to use formula (5.7) for the transformants of $P(t)$.

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